Note on Symmetric Functions.

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In this paper new proofs and more definite formulations of two previous theorems are given. The following notation is introduced:

$$D^{(m)}_{m+n-1-i}\,\alpha^{*} = |\,i_m i_{m-1} \,\dots\dots\, i_1| = \begin{vmatrix} \alpha_1^{i_m} & \alpha_1^{i_{m-1}} & \dots\dots & \alpha_1^{i_1} \\ \alpha_2^{i_m} & \alpha_2^{i_{m-1}} & \dots\dots & \alpha_2^{i_1} \\ \dots & \dots & \dots & \dots \\ \alpha_m^{i_m} & \alpha_m^{i_{m-1}} & \dots\dots & \alpha_m^{i_1} \end{vmatrix} \tag{1}$$

$$\Delta_x^{(m)}\,a^{\dagger} = \{x_1 x_2 \,\dots\dots\, x_m\} = \begin{vmatrix} a_{x_1} & a_{x_2} & \dots\dots & a_{x_m} \\ a_{x_1-1} & a_{x_2-1} & \dots\dots & a_{x_m-1} \\ \dots & \dots & \dots & \dots \\ a_{x_1-m+1} & a_{x_2-m+1} & \dots\dots & a_{x_m-m+1} \end{vmatrix}, \tag{2}$$

$$\begin{aligned} \begin{Bmatrix} p_1 p_2 \,\dots\dots\, p_m \\ x_1 x_2 \,\dots\dots\, x_m \end{Bmatrix} &= \text{the coefficient of } a_{p_1} a_{p_2} \dots\dots a_{p_m} \text{ in } \{x_1 x_2 \dots\dots x_m\} \\ &= \text{“} \qquad \text{“} \qquad \text{of } b_{p_1} b_{p_2} \dots\dots b_{p_m} \text{ in } \Delta_x^{(m)} b. \end{aligned} \tag{3}$$

§1.—*Theorem* I.

The product of a symmetric function $\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\dots\dots\alpha_m^{p_m}$ by the alternant $|0, 1, 2, \dots\dots m-1|$ is obtained by adding the p's in all possible permutations to the exponents of the columns of the alternant written as a determinant in which each line contains the powers of a single letter, thus giving the product in the general case as the sum of $m\,!$ alternants.

* These subscript indices are abbreviations; written in full, they would contain all the i's and all the x's respectively. For restrictions upon these and other notations, see §7. † Ibid.

§2.—*Proofs of Theorem* I.

1. Muir has covered this theorem in proving a similar proposition for the product of the alternant $|q_1 q_2 \ldots . q_m|$ and the preceding symmetric function from considerations of symmetry, and the fact that the product as a whole must be an alternating function.*

2. The writer has also given a proof, based on substitutions, in which it is shown that the substitutions,

$$\begin{pmatrix} 1 & 2 & 3 & \ldots . & m \\ i_1 & i_2 & i_3 & \ldots . & i_m \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & \ldots . & m \\ j_1 & j_2 & j_3 & \ldots . & j_m \end{pmatrix} \text{ and } \begin{pmatrix} j_1 & j_2 & j_3 & \ldots . & j_m \\ i_1 & i_2 & i_3 & \ldots . & i_m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 & \ldots . & j_m \\ 1 & 2 & 3 & \ldots . & m \end{pmatrix},$$

when applied to the straightforward multiplication of the alternant and symmetric function, and that demanded by the theorem respectively, give the same term, and hence give the required $(m!)^2$ terms identically the same.† This proof would apply without change if the alternant $|q_1 q_2 \ldots . q_m|$ had been used.

3. Professor W. H. Metzler has suggested the following proof: If we multiply the alternant $|q_1 q_2 \ldots . q_m|$ by $s_p = \alpha_1^p + \alpha_2^p + \ldots . \alpha_m^p$, we get, from a well-known theorem in determinants,

$$\begin{aligned} &(\alpha_1^{p_1} + \alpha_2^{p_2} + \ldots . \alpha_m^{p_m}) \times |q_1 q_2 \ldots . q_m| \\ &\quad = |p+q_1, q_2, \ldots . q_m| + |q_1, p+q_2, \ldots . q_m| + \ldots . + |q_1, q_2, \ldots . p+q_m|. \qquad (4) \end{aligned}$$

As every symmetric function of the roots can be expressed as a function of $s_1, s_2, \ldots . s_m$, the symmetric function $\Sigma \alpha_1^{p_1} \alpha_2^{p_2} \ldots . \alpha_m^{p_m}$, by which we are to multiply our alternant, can be expressed as

$$\phi(s_1, s_2, \ldots . s_m) \equiv s_{p_1} s_{p_2} \ldots . s_{p_m} + A \Sigma s_{p_1} s_{p_2} \ldots . s_{p_{m-1}+p_m} + \ldots . \qquad (5)$$

Now, from the known properties of the coefficients in ϕ,‡ it is easily seen that in the product of our alternant by ϕ, every alternant of the form

* Muir, "Determinants." 1882, p. 176, §129.

† American Mathematical Monthly, Vol. 6 (1899), p. 25. The author there attributed theorems I and II to Professor Gordan. Professor Metzler has kindly called the writer's attention to the reference to Muir, from which it appears that Muir has the priority of publication as far at least as theorem I is concerned. It may, however, be added that in a recent letter Professor Gordan states he has used the two theorems for the last thirty years.

‡ For the exact form of the A's, see Faà di Bruno, "Binäre Formen," p. 8, or Am. Math. Monthly, Vol. 5 (1898), p. 164, or Vol. 7 (1900), p. 66.

$|q_1 + x_1, q_2 + x_2, \ldots . q_m + x_m|$, where $x_1, x_2, \ldots . x_m$ are not some permutation of the p's, will have zero as coefficient.

§3.—*Theorem* II.

From theorem I another may be obtained by eliminating the α's in the right member, interpreted as the roots of

$$f(x) = a_0 x^m + a_1 x^{m-1} + \ldots . + a_m = 0, \tag{6}$$

by means of the theorem of corresponding matrices,* which expresses the symmetric function $a_0^n \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \ldots . \alpha_m^{p_m}$, as a sum of determinants of the n^{th} order in the a's.† This we shall call theorem II.‡ The developments of the next section show, however, *that theorem II can be proved independently of theorem I, and that theorem I can be made to depend upon theorem II.*

§4.—*More Exact Formulation of Theorems* I *and* II.

In order to throw out the numerous permutations which occur in the preceding proofs, i. e. *in order to deal with combinations only instead of permutations,* and to define the coefficient of any one alternant in the α's or determinant in the a's, according as theorem I or theorem II is desired, the following method may be used:

If
$$\phi(x) = b_0 x^n + b_1 x^{n-1} + \ldots . + b_n, \tag{7}$$
then of the matrices

$$\begin{vmatrix} a_0 & a_1 & \ldots . & a_m & 0 & \ldots . & 0 \\ 0 & a_0 & \ldots . & & a_m & 0 \ldots . & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_0 & & & \ldots . & a_m \end{vmatrix} = A, \ n \text{ lines}, \tag{8}$$

$$\begin{vmatrix} b_0 & b_1 & \ldots . & b_n & 0 & \ldots . & 0 \\ 0 & b_0 & \ldots . & & b_n & 0 \ldots . & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots . & 0 & b_0 & & \ldots . & b_n \end{vmatrix} = B, \ m \text{ lines}, \tag{9}$$

$$\begin{vmatrix} \alpha_1^{m+n-1} & \alpha_1^{m+n-2} & \ldots . & 1 \\ \alpha_2^{m+n-1} & \alpha_2^{m+n-2} & \ldots . & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_m^{m+n-1} & \alpha_m^{m+n-2} & \ldots . & 1 \end{vmatrix} = C, \ m \text{ lines}, \tag{10}$$

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* Gordan, "Invariantentheorie," Vol. 1, p. 95.

† Am. Math. Monthly, Vol. 6 (1899), p. 2.

‡ In the writer's former paper (l. c.) both theorems were treated practically as one.

A and C are corresponding matrices, and $\left|\begin{matrix}A\\B\end{matrix}\right|$ is the Sylvestrian determinant resultant of f and ϕ. By Laplace's development,

$$R_{f,\phi} = \left|\begin{matrix}A\\B\end{matrix}\right| = \Sigma(-1)_\mu \Delta_x^{(n)} a \Delta_i^{(m)} b.^* \tag{11}$$

By the theorem of corresponding matrices,

$$\Delta_x^{(n)} a = (-1)^\mu \lambda D_i^{(m)} \alpha, \tag{12}$$

whence†

$$R_{f,\phi} = \lambda\Sigma D_i^{(m)} \alpha \Delta_i^{(m)} b = \lambda \Sigma D_{m+n-1-i}^{(m)} \alpha \Delta_{m+n-1-i}^{(m)} b. \tag{13}$$

In each determinant $\Delta_{m+n-1-i}^{(m)} b$ we may pick out the term containing $b_{n-p_1} b_{n-p_2} \ldots . b_{n-p}$, and we may thus rearrange our sum with reference to terms of this form and write

$$\begin{aligned} R_{f,\phi} = \lambda \Sigma b_{n-p_1} b_{n-p_2} \ldots . b_{n-p_m} \\ \times \Sigma \begin{Bmatrix} n-p_1\, n-p_2 \ldots . n-p_m \\ m+n-1-i_m \ldots . m+n-1-i_1 \end{Bmatrix} |i_m i_{m-1} \ldots . i_1|. \end{aligned} \tag{14}$$

Again,

$$\begin{aligned} R_{f,\phi} &= a_0^n (b_0\alpha_1^n + b_1\alpha_1^{n-1} + \ldots b_n)(b_0\alpha_2^n + b_1\alpha_2^{n-1} + \ldots b_n) \ldots (b_0\alpha_m^n + b_1\alpha_m^{n-1} + \ldots b_n) \\ &= a_0^n \Sigma b_{n-p_1} b_{n-p_2} \ldots . b_{n-p_m} \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \ldots . \alpha_m^{p_m}. \end{aligned} \tag{15}$$

By equating the coefficients of $b_{n-p_1} b_{n-p_2} \ldots . b_{n-p_m}$ in both developments of the resultant, we have

$$\begin{aligned} a_0^n \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \ldots . \alpha_m^{p_m} \\ = \lambda \Sigma \begin{Bmatrix} n-p_1\, n-p_2 \ldots . n-p_m \\ m+n-1-i_m \ldots . m+n-1-i_1 \end{Bmatrix} |i_m i_{m-1} \ldots . i_1|. \end{aligned} \tag{16}$$

By corresponding matrices,

$$a_0^n = \lambda | m-1,\ m-2, \ldots . 1,\ 0|; \tag{17}$$

also, we have

$$|m-1,\ m-2, \ldots . 1, 0| = (-1)^{\frac{m(m-1)}{2}} |0,\ 1,\ 2, \ldots . m-1|; \tag{18}$$

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* The subscript complexes of indices x and i together make up all the indices $0, 1, 2, \ldots m+n-1$, i. e. these determinants contain no pair of corresponding columns from the two matrices. Also any a, with an index less than 0 or greater than m, is zero; similarly for the b's. See §7.

† Compare Gordan, "Invariantentheorie," Vol. 1, p. 184.

similarly with $|i_m i_{m-1} \dots\dots i_1|$; and we have in succession

$$\begin{Bmatrix} n-p_1 & \dots\dots n-p_m \\ m+n-1-i_m & \dots\dots m+n-1-i_1 \end{Bmatrix} = (-1)^{\frac{m(m-1)}{2}} \begin{Bmatrix} p_1 p_2 & \dots\dots p_m \\ i_m i_{m-1} & \dots\dots i_1 \end{Bmatrix}$$
$$= (-1)^{\frac{m(m-1)}{2}+\frac{m(m-1)}{2}} \begin{Bmatrix} p_1 p_2 & \dots\dots p_m \\ i_1\ i_2 & \dots\dots i_m \end{Bmatrix} = \begin{Bmatrix} p_1 p_2 & \dots\dots p_m \\ i_1\ i_2 & \dots\dots i_m \end{Bmatrix} \qquad (19)$$

by taking first the complements of the indices with respect to n and then double transpositions of the elements of the determinant

$$\begin{vmatrix} b_{i\ -m+1} & \dots\dots b_{i_1-m+1} \\ b_{i_m-m+2} & \dots\dots b_{i_1-m+2} \\ \dots\dots & \dots\dots \\ b_{i_m} & \dots\dots b_{i_1} \end{vmatrix}.$$

Using in (16) the values obtained in (17), (18) and (19), we have, as the expression of theorem I,

$$1.\ |0, 1, 2, \dots\dots m-1|\, \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \dots\dots \alpha_m^{p_m} = \Sigma \begin{Bmatrix} p_1 p_2 & \dots\dots p_m \\ i_1\ i_2 & \dots\dots i_m \end{Bmatrix} |i_1 i_2 \dots\dots i_m|. \qquad (20)$$

By using the theorem of corresponding matrices in (16), *or by directly expanding* (11) *after the manner of* (14), we obtain the expression for theorem II,

$$2.\ \ \alpha_0^n \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \dots\dots \alpha_m^{p_m} = \Sigma (-1)^\mu \begin{Bmatrix} p_1 p_2 & \dots\dots p_m \\ i_1\ i_2 & \dots\dots i_m \end{Bmatrix} \{ i_{\ +1}\, i_{m+2} \dots\dots i_{m+n} \}, \qquad (21)$$

which expresses a symmetric function of the α's homogeneously as a sum of determinants of the a's of the n^{th} order.

§5.—*The Coefficients* $\begin{pmatrix} q_1 q_2 \dots\dots q_n \\ 0^n p_1 p_2 \dots\dots p_m \end{pmatrix}$* *in Terms of the Coefficients* $\begin{Bmatrix} p_1 p_2 & \dots\dots p_m \\ i_1\ i_2 & \dots\dots i_m \end{Bmatrix}$.

If we expand the right member of (21) and collect the coefficient of the term $a_{q_1} a_{q_2} \dots\dots a_{q_n}$, which we denote by $\begin{pmatrix} q_1 q_2 \dots\dots q_n \\ 0^n p_1 p_2 \dots\dots p_m \end{pmatrix}$ according to notation previously used elsewhere,† we have, since we shall show in §7 that μ is

* The author's dissertation, "Die Entwickelung der Sylvester'schen Determinante nach Normal-Formen." Leipzig, B. G. Teubner, 1898, pp. 4 and 39, and Am. Math. Monthly, Vol. 6 (1899), pp. 55, 57, 104 et seq.

† Ibid.

constant and equal to $p_1+p_2+\ldots\ldots p_m$,

$$\binom{\quad q_1 q_2 \ldots\ldots q_n}{0^n p_1 p_2 \ldots\ldots p_m}$$
$$=(-1)^{p_1+p_2+\ldots\ldots p_m}\Sigma\begin{Bmatrix}p_1 p_2 & \ldots\ldots & p_m\\ i_1\ i_2 & \ldots\ldots & i_m\end{Bmatrix}\begin{Bmatrix}q_1 q_2 & \ldots\ldots & q_n\\ i_{m+1}\, i_{m+2} & \ldots\ldots & i_{m+n}\end{Bmatrix}. \tag{22}$$

Hence (21) becomes

$$a_0^n\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_m^{p_m}=\Sigma\binom{\quad q_1 q_2 \ldots\ldots q_n}{0^n p_1 p_2 \ldots\ldots p_m}a_{q_1}a_{q_2}\ldots\ldots a_{q_n}$$
$$=(-1)^{p_1+p_2+\ldots\ldots p_m}\Sigma\Sigma\begin{Bmatrix}p_1 p_2 & \ldots\ldots & p_m\\ i_1\ i_2 & \ldots\ldots & i_m\end{Bmatrix}\begin{Bmatrix}q_1 q_2 & \ldots\ldots & q_n\\ i_{m+1} & \ldots\ldots & i_{m+n}\end{Bmatrix}a_{q_1}a_q\ldots\ldots a_{q_n}. \tag{23}$$

§6.—*The Coefficients* $q_1 q_2\ldots\ldots q_n|n-p_1\, n-p_2\ldots\ldots n-p_m$* *in Terms of the Coefficients* $\begin{Bmatrix}p_1 p_2 & \ldots\ldots & p_m\\ i_1\ i_2 & \ldots\ldots & i_m\end{Bmatrix}$.

If we substitute the value of the symmetric function, as given by (21), in the development of the resultant as expressed by (15), and collect the coefficient of $a_{q_1}a_{q_2}\ldots\ldots a_{q_n}b_{n-p_1}b_{n-p_2}\ldots\ldots b_{n-p_m}$, which has also been previously denoted elsewhere† by $q_1 q_1\ldots\ldots q_n|n-p_1\, n-p_2\ldots\ldots n-p_m$, we have

$$q_1 q_2\ldots\ldots q_n|n-p_1\, n-p_2\ldots\ldots n-p_n$$
$$=(-1)^{p_1+p_2+\ldots\ldots p_m}\Sigma\begin{Bmatrix}p_1 p_2 & \ldots\ldots & p_m\\ i_1\ i_2 & \ldots\ldots & i_m\end{Bmatrix}\begin{Bmatrix}q_1 q_2 & \ldots\ldots & q_n\\ i_{m+1}\, i_{m+2} & \ldots\ldots & i_{m+n}\end{Bmatrix}, \tag{24}$$

and, by using this value, (15) becomes

$$R_{f,\phi}=(-1)^{p_1+p_2+\ldots\ldots p_m}\Sigma\Sigma\Sigma\begin{Bmatrix}p_1 p_2 & \ldots\ldots & p_m\\ i_1\ i_2 & \ldots\ldots & i_m\end{Bmatrix}\begin{Bmatrix}q_1 q_2 & \ldots\ldots & q_n\\ i_{m+1} & \ldots\ldots & i_{m+n}\end{Bmatrix}$$
$$\times a_{\ 1}a_{q_2}\ldots\ldots a_{q_n}b_{n-p_1}b_{n-p_2}\ldots\ldots b_{n-p_m}. \tag{25}$$

§7.—*Restrictive Relations.*

The foregoing summations are restricted by the following conditions on the indices and exponents. For (20) and (21):‡

* The author's dissertation, "Die Entwickelung der Sylvester'schen Determinante nach Normal-Formen." Leipzig. B. G. Teubner, 1898, pp. 4 and 39, and Am. Math. Monthly, Vol. 6 (1899), pp. 55, 57, 104 et seq.

† Ibid.

‡ For (22) and (23) $q_1+q_2+\ldots\ldots q_n=p_1+p_2+\ldots\ldots p_m$, and for (23) $mn\geqq p_1+p_2+\ldots p_m\geqq 0$.

$$n \geqq p_1 \geqq p_2 \geqq \ldots\ldots \geqq p_m \geqq 0, \tag{26}$$

$$p_1 + m - 1 \geqq i_m > i_{m-1} > \ldots\ldots > i_1 \geqq 0, \tag{27}$$

$$i_{m+n} > i_{m+n-1} > \ldots\ldots > i_{m+1}, \tag{28}$$

$$i_1 + i_2 + \ldots\ldots i_m = \frac{m(m-1)}{2} + p_1 + p_2 + \ldots\ldots p_m. \tag{29}$$

$i_{m+1}, i_{m+2}, \ldots\ldots i_{m+n}$ are the indices of the elements of the first line of the determinant corresponding to $|i_m i_{m-1} \ldots\ldots i_1|$. The distinction of a and b may be dropped in (20) and (21) and then with respect to

$$\begin{Bmatrix} p_1 p_2 & \ldots\ldots & p_m \\ i_1\ i_2 & \ldots\ldots & i_m \end{Bmatrix}, \quad a_{n+i} = a_{-i} = 0;$$

with respect to

$$\{i_{m+1} i_{m+2} \ldots\ldots i_{m+n}\}, \quad a_{m+i} = a_{-i} = 0. \tag{30}$$

Since the sum of the indices 0, 1, 2, ... $m + n - 1$, each increased by 1 is $\dfrac{(m+n)(m+n+1)}{2}$, we have $m + n - i_m + m + n - i_{m-1} + \ldots\ldots m + n - i_1$ $+ i_{m+1} + 1 + \ldots\ldots + i_{m+n} + 1 = \dfrac{(m+n)(m+n+1)}{2}$, or

$$i_{m+1} + \ldots\ldots i_{m+n} - \frac{n(n-1)}{2} = i_1 + i_2 + \ldots\ldots i_m - \frac{m(m-1)}{2}$$
$$= (\text{by } (29))\ p_1 + p_2 + \ldots\ldots p_m. \tag{31}$$

Now

$$\mu = (i_{m+1} + 1) - 1 + (i_{m+2} + 1) - 2 \ldots\ldots (i_{m+n} + 1) - n$$
$$= i_{m+1} + i_{m+2} + \ldots\ldots i_{m+n} - \frac{n(n-1)}{2}, \tag{32}$$

or by (31) and (32),

$$\mu = p_1 + p_2 + \ldots\ldots p_m. \tag{33}$$

§8.—*The Calculation of the Coefficient*

$$\begin{Bmatrix} p_1 p_2 & \ldots\ldots & p_m \\ i_1\ i_2 & \ldots\ldots & i_m \end{Bmatrix}.$$

A recurrence formula for the calculation of $\begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1\, i_2 \dots i_m \end{Bmatrix}$ is already involved in the nature of the coefficient as expressed. If r numbers be common to the two series $p_1, p_2, \dots p_m$ and $i_1, i_2, \dots i_m$, so that

$$\begin{aligned} i_{\lambda_1} &= p_{\lambda_1}, \\ i_{\lambda_2} &= p_{\lambda_2}, \\ &\dots\dots \\ i_{\lambda_r} &= p_{\lambda_r}, \end{aligned} \tag{34}$$

we have, by expanding the determinant $\{i_1\, i_2 \dots i_m\}$ in terms of the elements of the first line,

$$\begin{Bmatrix} p_1 p_2 \dots p_m \\ i_1\, i_2 \dots i_m \end{Bmatrix} = \sum_{\kappa=1}^{\kappa=r} (-1)^{\lambda_\kappa - 1} \begin{Bmatrix} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_{\kappa-1}} p_{\lambda_{\kappa+1}} \dots p_{\lambda_m} \\ i_1 - 1,\, i_2 - 1, \dots i_{\lambda_{\kappa-1}} - 1,\, i_{\lambda_{\kappa+1}} - 1 \dots i_m - 1 \end{Bmatrix}, \tag{35}$$

a coefficient of order m, expressed as a sum of several of order $m-1$. If no numbers of the two series are common, the coefficient is zero. If a lower index becomes negative, the coefficient is also zero. The order of the upper indices is indifferent for calculation. It is obvious that $\begin{Bmatrix} x \\ x \end{Bmatrix} = 1$.

§9.—*Examples.*

1. $\begin{Bmatrix} 012 \\ 123 \end{Bmatrix} = \begin{Bmatrix} 02 \\ 12 \end{Bmatrix} - \begin{Bmatrix} 01 \\ 02 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = -2.$
2. $\begin{Bmatrix} 012 \\ 024 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 13 \end{Bmatrix} - \begin{Bmatrix} 01 \\ -13 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} = 1.$
3. $\begin{Bmatrix} 0^2 13 \\ 0127 \end{Bmatrix} = \begin{Bmatrix} 013 \\ 016 \end{Bmatrix} - \begin{Bmatrix} 0^2 3 \\ -116 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 05 \end{Bmatrix} - \begin{Bmatrix} 03 \\ -15 \end{Bmatrix} = 0.$
4. $\begin{Bmatrix} 0123 \\ 0345 \end{Bmatrix} = \begin{Bmatrix} 123 \\ 234 \end{Bmatrix} - \begin{Bmatrix} 012 \\ -134 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 23 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 13 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} = -2.$
5. $\begin{Bmatrix} 0123 \\ 0156 \end{Bmatrix} = 0.$

Since, by (31) and (33), the equation

$$\dot{i}_{m+1} + \dot{i}_{m+2} + \dots \dot{i}_{m+n} - \frac{n(n-1)}{2} = \mu = p_1 + p_2 + \dots p_m$$

exists, or

$$\dot{i}_{m+1} + \dot{i}_{m+2} - 1 + \dot{i}_{m+3} - 2 + \dots \dot{i}_{m+n} - (n-1) = \mu, \qquad (36)$$

i. e., since the sum of the principal diagonal indices of the determinant $\{\dot{i}_{m+1} i_{m+2} \dots \dot{i}_{m+n}\}$ is equal to the weight of the symmetric function, a fact which we also know otherwise from general theory, it is best in practice to form the different sets of principal diagonal indices of the determinants $\{\dot{i}_{m+1} i_{m+2} \dots \dot{i}_{m+n}\}$ first, as these are the sets most easily formed; by the addition of 0, 1, 2, ... $n-1$ respectively to these, we get the first line indices; we take next the complements of these with respect to $m+n-1$, and lastly, the remaining indices of the set 0, 1, 2 .... $m+n-1$, not found among the complements for the indices of the alternants, and lastly, write these indices under the indices of the symmetric function for the coefficient of the alternant and determinant when μ is even, but the negative of this coefficient for the coefficient of the determinant when μ is odd. It is also necessary to take $m = \mu$ in order to get a general result, while n is taken as the order of the symmetric function.

6. It is required to find $|01234|\Sigma\alpha_1^3\alpha_2^2$ and $\alpha_0^3\Sigma\alpha_1^3\alpha_2^2$. We have the following calculation:

Principal Diagonal Indices. Sum $=\mu=5$.	First Line Indices. Sum $=\mu + \frac{n(n-1)}{2} = 8$.	Complements with respect to $m+n-1$. Sum $= mn + \frac{n(n-1)}{2} - \mu = 13$.	Remaining Indices. Sum $= \frac{m(m-1)}{2} + \mu = 15$.	Coefficients.
$\lambda_1, \lambda_2, \lambda_3$	$\lambda_1, \lambda_2+1, \lambda_3+2$	$\lambda-(m+n-1)$		
005	017	067	12345	$\left\{\begin{matrix} 0^3 23 \\ 12345 \end{matrix}\right\}$
014	026	157	02346	$\left\{\begin{matrix} 0^3 23 \\ 02346 \end{matrix}\right\}$
113	125	256	01347	$\left\{\begin{matrix} 0^3 23 \\ 01347 \end{matrix}\right\}$
122	134	346	01257	$\left\{\begin{matrix} 0^3 23 \\ 01257 \end{matrix}\right\}$
023	035	247	01356	$\left\{\begin{matrix} 0^3 23 \\ 01356 \end{matrix}\right\}$

The calculation of the coefficients follows:

$$\left\{\begin{matrix}0^3 23\\12345\end{matrix}\right\}=-\left\{\begin{matrix}0^3 3\\0234\end{matrix}\right\}+\left\{\begin{matrix}0^3 2\\0134\end{matrix}\right\}=-\left\{\begin{matrix}0^2 3\\123\end{matrix}\right\}+\left\{\begin{matrix}0^2 2\\023\end{matrix}\right\}$$

$$=-\left\{\begin{matrix}0^2\\01\end{matrix}\right\}+\left\{\begin{matrix}02\\12\end{matrix}\right\}=-\left\{\begin{matrix}0\\0\end{matrix}\right\}-\left\{\begin{matrix}0\\0\end{matrix}\right\}=-2.$$

$$\left\{\begin{matrix}0^3 23\\02346\end{matrix}\right\}=\left\{\begin{matrix}0^2 23\\1235\end{matrix}\right\}=-\left\{\begin{matrix}0^2 3\\024\end{matrix}\right\}+\left\{\begin{matrix}0^2 2\\014\end{matrix}\right\}=-\left\{\begin{matrix}03\\13\end{matrix}\right\}+\left\{\begin{matrix}02\\03\end{matrix}\right\}=+\left\{\begin{matrix}0\\0\end{matrix}\right\}+\left\{\begin{matrix}2\\2\end{matrix}\right\}=2.$$

$$\left\{\begin{matrix}0^3 23\\01347\end{matrix}\right\}=\left\{\begin{matrix}0^2 23\\0236\end{matrix}\right\}=\left\{\begin{matrix}023\\125\end{matrix}\right\}=-\left\{\begin{matrix}03\\04\end{matrix}\right\}=-\left\{\begin{matrix}3\\3\end{matrix}\right\}=-1.$$

$$\left\{\begin{matrix}0^3 23\\01257\end{matrix}\right\}=\left\{\begin{matrix}0^2 23\\0146\end{matrix}\right\}=\left\{\begin{matrix}023\\035\end{matrix}\right\}=\left\{\begin{matrix}23\\24\end{matrix}\right\}=\left\{\begin{matrix}3\\3\end{matrix}\right\}=1.$$

$$\left\{\begin{matrix}0^3 23\\01356\end{matrix}\right\}=\left\{\begin{matrix}0^2 23\\0245\end{matrix}\right\}=\left\{\begin{matrix}023\\134\end{matrix}\right\}=-\left\{\begin{matrix}02\\03\end{matrix}\right\}=-1.$$

By substituting the values of the preceding coefficients, we have, as the results of the calculation, and as illustrations of theorems I and II (Formulas (20) and (21)),

$$|01234|\Sigma\alpha_1^3\alpha_2^2=-2|12345|+2|02346|-|01347|+|01257|-|01356|, \qquad \text{I.}$$

$$a_0^3\,\Sigma\alpha_1^3\alpha_2^2=2\begin{vmatrix}a_0 & a_1 & 0\\0 & a_0 & 0\\0 & 0 & a_5\end{vmatrix}-2\begin{vmatrix}a_0 & a_2 & 0\\0 & a_1 & a_5\\0 & a_0 & a_4\end{vmatrix}+\begin{vmatrix}a_1 & a_2 & a_5\\a_0 & a_1 & a_4\\0 & a_0 & a_3\end{vmatrix}$$

$$-\begin{vmatrix}a_1 & a_3 & a_4\\a_0 & a_2 & a_3\\0 & a_1 & a_2\end{vmatrix}+\begin{vmatrix}a_0 & a_3 & a_5\\0 & a_2 & a_4\\0 & a_1 & a_3\end{vmatrix}. \qquad \text{II.}$$

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